UPSC Mathematics Optional

Previous Year Question Solutions

COMPLEX ANALYSIS

2015

Q1. Show that the function $v(x,y)=\ln\left(x^2+y^2\right)+x+y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also, find the corresponding analytic function $f(z)=4+iv$ in terms of z.

[10 Marks]

Solution:

Given, $v(x,y)=\ln\left(x^2+y^2\right)+x+y$

$$\frac{\partial V}{\partial x}=\frac{2x}{x^2+y^2}+1$$

$$\frac{\partial^2 V}{\partial x^2}=\frac{2\left(x^2+y^2\right)-4x^2}{\left(x^2+y^2\right)^2}$$

$$=\frac{-2x^2+2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial V}{\partial y}=\frac{2y}{x^2+y^2}+1$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{2\left(x^2+y^2\right)-4y^2}{\left(x^2+y^2\right)^2}=\frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{-\left(2x^2-2y^2\right)}{\left(x^2+y^2\right)^2}+\frac{2x^2-2y^2}{\left(x^2+y^2\right)^2}=0$$

Hence it is a Harmonic function.

$$\frac{\partial v}{\partial y}=\frac{2y}{x^2+y^2}+1=\frac{\partial y}{\partial x}\Rightarrow\frac{\partial y}{\partial x}=\frac{2y}{x^2+y^2}+1$$

Integrating w.r.t. x.

$$u=\tan^{-1}\left(\frac{x}{y}\right)+x+f(y)$$

$$\frac{\partial u}{\partial y}=\frac{1}{\left(\left(1+\frac{x}{y}\right)^2\right)}+xy+f'(y)$$

$$\frac{\partial u}{\partial y}=\frac{y^2}{y^2+x^2}+xy+f'(y)$$

$$f'(y)=\frac{dy}{dy}-\frac{y^2}{y^2+x^2}-xy$$

$$f'(y) = \frac{2x}{x^2+y^2} + 1 - \frac{y^2}{y^2+x^2} - xy$$

$$f'(y) = \frac{2x-y^2}{x^2+y^2} + 1 \cdot xy$$

$$f(y) = \tan^{-1}\left(\frac{y}{x}\right) + y - xy^2 - \int\left[\frac{x^2+y^2}{x^2+y^2} - \frac{x^2}{x^2+y^2}\right]$$

$$f(y) = \tan^{-1}\left(\frac{y}{x}\right) + y - xy^2 - y + \frac{x}{2}\tan^{-1}\left(\frac{y}{x}\right)$$

$$f(y) = \tan^{-1}\left(\frac{y}{x}\right) - xy^2 + \frac{x}{2}\tan^{-1}\left(\frac{y}{x}\right)$$

Hence becomes $u = \tan^{-1}\left(\frac{y}{x}\right) + x + \tan^{-1}(y/x) + \frac{x}{2}\tan^{-1}\frac{y}{x} - xy^2$

$$u = 2\tan^{-1}\left(\frac{y}{x}\right) + \frac{x}{2}\tan^{-1}\left(\frac{y}{x}\right) + x\left(1-y^2\right)$$

$$y = \tan^{-1}\left(\frac{y}{x}\right)\left[2 + \frac{x}{2}\right] + x\left(1-y^2\right)$$

$$f = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \frac{2y}{x^2+y^2} + 1 + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

Put $x = z, y = 0$

$$f'(z) = 1 + i\left(\frac{2z}{z^2} + 1\right)$$

$$= 1 + i\left(\frac{2}{z} + 1\right)$$

$$f(z) = z + i(2\ln z + z) \pm c$$

$$f(z) = z + i(z + \ln z^2) + c$$

Q2. Find all Possible Taylor's and Law rant's shies expansion of function $f(z) = \frac{2z-3}{z^2-3z+z}$ about the point $z = 0$

Solution:

$$-f(z) = \frac{2z-3}{z^2-3z+z} = \frac{2z-3}{(z-1)(z-2)}$$

By using partial paction

$$\frac{2z-3}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$= A(z-2) + B(z-1)$$

$\Rightarrow A + B = 2$ and $2A + B = 3$

$\therefore$ By Solving, we get,

$$A = 1, \ B = 1$$

$$\therefore \quad f(z) = \frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$f_1(z) = \frac{1}{(z-1)} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$f_1(z) = \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

Since $\left|\frac{1}{z}\right| < 1$ thence $|z| > 1$, so valid for $| = 1 > 1$.

Again, $f_2(z) = \frac{1}{(z-2)} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$

$$f_2 = \frac{1}{z}\left[1 + \frac{2}{z} + \frac{4}{z^2} + \cdots\right]$$

$$= \left[\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \cdots\right]$$

Since, $\left|\frac{2}{z}\right| < 1$: Hence valid for $|z| > 2$

Now, the Laurent's series

$$\left(f_1 + f_2\right)z = \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] + \left[\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \cdots - \cdots\right]$$

$\left(f_1 + f_2\right)z = \frac{2}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{9}{z^4} + \cdots$ is valid for $|z| > 2$ is the required Laurent's series.

for Taylor's series

$$f_1(z) = \frac{1}{(z-1)} = \frac{-1}{(1-z)} = -1(1-z)^{-1}$$

$$f_1(z) = -\left(1 + z + z^2 + z^3 + \cdots\right), |z| > 2.$$

$$f_2(z) = \frac{1}{(z-2)} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$f_2(z) = \frac{-1}{2}\left[1+\frac{z}{2}+\frac{z^3}{4}+\frac{z^3}{8}+\cdots\right]_{|z|>2}$$

$$\therefore (f_1+f_2)z = -1\left(1+z+z^2+\cdots\right)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\right)$$

$$(f_1+f_2)z = -\left[\left(1+z+z^2+z^3+\cdots\right)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)\right]$$

is the required Taylor's series.

Q3. State Cauchy's theorem. Using it evaluate the integral $\int_c \frac{e^z+1}{z(z+1)(z-i)^2}dz \quad c:|z|=2$

Solution:

Given, $\int_c \frac{e^z+1}{z(z+1)(z-i)^2}dz$

over the region $c:|z|=2$

$$f(z) = \frac{e^z+1}{z(z+1)(z-i)^2}$$

Poles are $z=0, z=-1$. of order 1.

$z=i$ of order 2

Now, residue at $z=0$.

$$\Rightarrow \lim_{z\to 0} z\cdot f(z) = \lim_{z\to 0} z\frac{e^z+1}{z(z+1)(z-i)}$$

$$= \frac{e^0+1}{(0+1)(0-i)^2} = \frac{2}{i^2} = -2.$$

Residue at $z=-1$

$$\Rightarrow \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to -1}\cancel{(z+1)}\cdot\frac{e^{z+1}}{z\,\cancel{(z+1)}\,(z-i)^2}$$

$$= \frac{e^{-1}+1}{(-1)(-1-i)^2}$$

$$= \frac{-\left(e^{-1}+1\right)}{(1+i)^2}$$

Residue at $z=i$ of order 2

$$\Rightarrow \lim_{z\to i}\frac{d}{dz}(z-i)^2\cdot f(z)$$

$$= \lim_{z\to i}\frac{d}{dz}\frac{\cancel{(z-i)^2}\cdot e^z+1}{z(z+1)\,\cancel{(z-1)^2}}$$

$$=\lim_{z\to i}\frac{d}{dz}\frac{e^z+1}{z(z+1)}$$

$$=\lim_{z\to i}\frac{e^z\left(z^2+z\right)-\left(e^2+1\right)(zz+1)}{z^2(z+1)^2}$$

$$=\lim_{z\to i}\frac{e^z\cdot z^2+e^z\cdot z-\left[2e^z\cdot z+e^z+2z+1\right]}{z^2[z+1]^2}$$

$$=\lim_{z\to i}\frac{e^z\left[z^2+z-2z-1\right]-2z-1}{z^2(z+1)^2}$$

$$=\lim_{z\to i}\frac{e^z\left[z^2-z-1\right]-2z-1}{z^2(z+1)^2}$$

$$=\frac{e^i\left(i^2-i-1\right)-2i-1}{i^2(1+i)^2}$$

$$=\frac{-\left(2e^i+2i+1+e^i\right)}{(-1)(1+i)^2}$$

$$=\frac{2e^i+2i+1+ie^i}{(1+i)^2}$$

$$=\frac{e^i(z+i)+(2i+1)}{(1+i)^2}$$

$\therefore\oint f(z)dz=2\pi i$ [residue at $z=0+$ Residue at $z=-1$ + residue at $z=i$]

$$=2\pi i\left[-2-\frac{\left(e^{-1}+1\right)}{(1+i)^2}+\frac{e^i(2+i)+(2i+1)}{(1+i)^2}\right]$$

$$\oint_c f(z)dz=-2\pi i\left[\frac{2+\left(e^{-1}+1\right)+e^{-1}(2+i)-2i-1}{(1+i)^2}\right]$$

$$\oint_c f(z)dz=-2\pi i\left[2+\left[\frac{e^{-1}-e^{-i}(2+i)-2i}{(1+i)^2}\right]\right.$$

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